The Mixed Poynting Vector

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We are very familiar with the concepts of specular transmittance and reflectance that are of fundamental importance in the characterization of our optical coatings. They are dimensionless quantities expressing the fraction of the incident light that is respectively transmitted through or reflected by a structure. We imagine the incident light to be monochromatic and the surfaces concerned to be flat and featureless so as to eliminate any incoherent scattering and so that the laws of reflection and refraction apply. Then, if the incident light is in the form of an infinite plane wave, the transmittance and reflectance are defined as the appropriate ratios of the normal components of irradiance of the transmitted and reflected waves to the normal irradiance of the incident wave. If the various light beams are confined in their areas then the ratios should be that of the total beam powers. Both these definitions assure unambiguously compatible results and their forms are necessary because of the refraction quantified by Snell’s Law.

There is, however, a problem. Our definition breaks down when the incident medium exhibits absorption. The problem is essentially a theoretical one, because the conditions under which a measurement of reflectance can actually be made will normally assure that the corresponding error be vanishingly small, but calculation should necessarily be correct. The most frequently adopted solution is to define reflectance in an incident medium free of absorption only.

The source of the problem is the mixed Poynting vector, the subject of this contribution. It is not a new concept, but, although now well understood, its history has been dogged by misunderstandings. Although usually mentioned solely in connection with the problem of reflectance and transmittance just mentioned, it is of much wider significance. So that the discussion does not become obscured by unnecessarily involved details, we shall limit the following discussion to optical waves propagating in the same or opposite direction, and, where applicable, normally incident on surfaces. Extension to more complicated cases should be reasonably clear.

The Poynting Vector

The instantaneous power density delivered by an electromagnetic wave is given by the Poynting vector, that is the cross, or vector, product of the electric and magnetic fields. This is, of course, a vector expression giving both the magnitude and the direction of the power flow. Since the units of electric field are volt metre$^{-1}$ and of magnetic field amp metre$^{-1}$, those of the power flow will be their product watt metre$^{-2}$.

\[ S = E \times H \]  

(1)

Following the original paper [1] we use the symbol $S$ for the vector. This is a nonlinear relationship where the real forms of the fields, rather than the complex, should be used. Since the product of two circular functions oscillates, it is the mean of this expression that yields the irradiance.

For a single linearly-polarized harmonic wave it is readily shown that the magnitude of the irradiance in a medium free from absorption is given by

\[ I = \frac{1}{2} \varepsilon_0 \omega^2 \varepsilon \left( \frac{1}{2} \gamma^2 + \frac{1}{2} \eta \varepsilon^2 \right) \]  

(2)

where $\varepsilon$ and $\Phi$ are the amplitudes, $\gamma$ the characteristic admittance, given by $\varepsilon_0 \eta$, where $\varepsilon_0$ is the admittance of free space and $\eta$, rather than the standard symbol $E$, is used to indicate irradiance.

We make much use of complex waves in our calculations but the mixing of the real and imaginary parts when multiplied as in (1) prevents their simple substitution in (1). We must therefore use the expression

\[ I = \frac{1}{2} \varepsilon_0 \omega^2 \varepsilon \left( \frac{1}{2} \gamma^2 + \frac{1}{2} \eta \varepsilon^2 \right) dt \]  

The first two terms combine to give a real expression because the second term is the complex conjugate of the first. The same is true for the combination of the third and fourth terms. But both $E$ and $H$ will have $\exp(i\omega t)$ as a factor together with a phase term that, although it may be a function of $z$, will be constant since $z$ is constant. The $\exp(i\omega t)$ part will therefore disappear in each of the first two terms yielding a constant, but in the third and fourth terms it will cause the real part of the summation to oscillate at twice the wave frequency to give an integral of zero. The complete result is then

\[ I = \frac{(EH^* + E^*H)}{4} = \frac{1}{2} \text{Re}(EH^*) \]  

(4)

This is the complex Poynting expression.

A Little History

John Henry Poynting (1852-1914) was a professor of physics at what is now the University of Birmingham, and, in 1884 he published the first of the papers [1] in which he derived an expression for the flow of energy in a propagating electromagnetic disturbance. Here is his description.

“On interpreting the expression it is found that it implies that the energy flows as stated before, that is, perpendicularly to the plane containing the lines of electric and magnetic force, that the amount crossing unit area per second of this plane is equal to the product of the electromotive intensity by the magnetic intensity by sine included angle

\[ I = \frac{1}{2} \text{Re}(EH^*) \]

while the direction of flow is given by the three quantities, electromotive intensity, magnetic intensity, flow of energy, being in right handed order.” (Equation unnumbered in original) It can be seen that this is equivalent to a vector product of electric and magnetic fields.

The two volumes of James Clerk Maxwell’s great work, A Treatise on Electricity and Magnetism [2], published in 1873, set out the physical theory that we still use today, but not quite in the same form. Instead of the four differential vector equations that, with their beautiful symmetry, are familiar to us, there were some twenty relationships. Maxwell used Hamilton’s quaternions rather than vectors, and so it was all much more difficult to assimilate. It is Oliver Heaviside (1850-1925) whom we must thank for the wonderfully compact form in which our theory is presented today.

Heaviside was born in London and attended school until the age of 16.
From then onwards he was largely self-educated. He was a nephew by marriage of Charles Wheatstone, the developer of the telegraph, and the only posts he ever held were as a telegraph operator, first in Denmark (he taught himself Danish) and then in Newcastle upon Tyne in the north east of England. After some six years he left employment completely and devoted himself to his own private studies. He made an enormous number of contributions in mathematics, physics, and engineering, the Kennelly-Heaviside Layer in the ionosphere is named after him, he is the inventor of coaxial cable, and he was elected to Fellowship of the Royal Society, but the work that interests us here, is his complete recasting of Maxwell’s Equations in the vector form in which we now know them. He also created the terminology that we use: words like impedance, admittance, and conductance, for example. His contributions were first published weekly as a series of articles in *The Electrician*, essentially a trade publication with some technical content. The modest payment for the articles was virtually his only source of income. The first series appeared in the 1880’s and were then reprinted as his book *Electrical Papers*, two volumes of which were published in 1892 [3].

His result for the flow of energy is exactly that of Poynting but expressed as a vector product:

\[ S = \frac{V \cdot H}{4\pi} \]  

(6)

The \( V \) in the expression is Heaviside’s notation for a vector product.

The \( 4\pi \) in the denominator of both the Poynting and Heaviside expressions is a matter of units. Heaviside had the following to say about the frequent occurrence of this factor:

“The excessence \( 4\pi \) is a mere question of units, and need not be discussed here. The \( 4\pi \)'s are particularly obnoxious and misleading in the theory of magnetism. Privately I use units which get rid of them completely, and then, for publication, liberally season with \( 4\pi \)'s to suit the taste of B.A. unit-fed readers.”

B.A. stands for the British Association for the Advancement of Science. Its 1862 committee had made recommendations, largely adopted, for a system of units, essentially the CGS system as a scientific set, together with a related set of practical units of more useful magnitude. The reason for the split between scientific and practical is understandable, but it has led to all sorts of difficulties, not least those of students. At last, from 1960 onwards, the SI system (named for *le Système International d’Unités*) succeeded in bringing the practical and scientific units together and removing most of the \( 4\pi \)'s, as in equation (1), although, because the term represents the internal solid angle of a sphere, it must occur somewhere in the system.

Both Poynting and Heaviside were interested more in the transport of alternating electrical energy in conductors used for telegraphy, and for electrical distribution, than in the irradiance of optical waves. Both established that the medium outside the conductor supported the flow of energy while the conductor itself exhibited a skin depth and was responsible for the ohmic power loss. This was considered preposterous by the practical engineers in the field who ignored inductance completely, and there was, for a time, bitter argument between those who supported the results derived using Maxwell’s theory and the practical community that believed that everything was confined inside the copper wire and that to carry greater power it simply had to be of greater diameter. Heaviside’s publications were, for a time, completely blocked in retaliation for his criticism of William Henry Preece, Chief Electrician of the Post Office and head of the British government telegraph service, and the powerful figure representing the practical community. Eventually, of course, the practical people had to give way because the effects and related problems actually existed, and by the early 1890’s the new theory was becoming largely accepted and Heaviside’s publications had resumed. For a fascinating account of the controversy see Hunt [4].

**Fundamentals**

We will keep to our quite simple arrangement that nevertheless illustrates the more general. In any combination of waves we limit ourselves to waves of identical frequency exhibiting no discontinuities of phase. We set up our usual set of characteristic admittance \( y \) and refractive index \( n - ik \), where \( y \) is given in its fundamental SI units rather than the usual free space units. We can assume that the wave is linearly polarized along the \( x \)-direction. Then we can write \( E_x \) and \( H_y \) for the electric and magnetic fields, where, in their complex forms they are given by

\[ E = E_x \exp(i(\omega t - kx)) \]

\[ H = H_y \exp(i(\omega t - kx)) \]

(7)

where \( k \) is \( 2\pi(n - ik)/\lambda \). Then, from (4), the irradiance is given by:

\[ I = \frac{1}{2} \text{Re}(E_x H_y^* + E_y H_x^*) = \frac{1}{2} \text{Re}(\phi)E_x^2 + \frac{1}{2} m^2E_y^2 \]

(8)

\( \phi \) being the admittance of free space and \( y \) being given by \((n - ik)\phi \). This is exactly the result of equation (2), but in (2) there was the assumption of zero absorption.

Now let there be two waves defined only as possessing such polarization and direction that the resultant fields are simply the sum of the individual fields. Then, denoting the fields of the waves by suffixes 1 and 2 we find the complex Poynting expression:

\[ I = \frac{1}{2} \text{Re}((E_1^* + E_2^*) (H_1 + H_2)^*) = \frac{1}{2} \text{Re}(E_1^* H_2 + E_2^* H_1) \]

\( I_1 + I_2 + \frac{1}{2} \text{Re}(E_1 H_2^* + E_2 H_1^*) \]

(9)

where we have treated the individual irradiances, \( I_1 \) and \( I_2 \), as we would were they propagating completely independently. The third term, involving the mixed products is the mixed Poynting expression, or mixed Poynting vector when written as a vector quantity, and, from now on, we shall refer to it as the latter, even when written in scalar terms. It must be admitted that the mixed Poynting vector is usually invoked when reflection at an interface is involved, and hence the waves are counterpropagating. However there is no good reason to limit the concept to one particular configuration.

Let us continue with the case where the waves are propagating in parallel, but for the moment we assume complete transparency, that is \( k \) is zero. We remember that each wave is of identical frequency, identical linear polarization, and propagating along the positive direction of the \( z \)-axis with relative phase \( \phi \). We keep the amplitudes real and hold the relative phase in the phase factors of the waves. From (9) we have

\[ I = I_1 + I_2 + \frac{1}{2} \text{Re}(E_x H_y + E_y H_x) \]

\( = I_1 + I_2 + \frac{1}{2} \text{Re}(\gamma E_x E_y + \gamma E_y E_x) \cos \phi \)

(10)

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It is clear that, in this case, the mixed Poynting vector is simply the interference term. In fact, when we are dealing with a mixture of waves of differing frequency and relative phase the bracketed part of the mixed Poynting vector and interference term in the second line of (10) becomes the mutual coherence function [5].

Now let us reverse wave 2 so that it propagates along the negative z-direction. The extinction coefficient is still zero. The phase factors become:

\[
\text{Wave 1: } \exp\left[\left(\alpha z - \alpha z\right)\right] \quad \text{Wave 2: } \exp\left[\left(\alpha z + \alpha z\right)\right]
\]

where \(\alpha\) is real. In addition to the change in sign in the phase factor we must also make sure that the Poynting vector for wave 2 points along the negative z-direction and we do that by flipping the direction of the magnetic field.

\[
I = I_1, \quad I_2, \quad I_3, \quad I_4
\]

where we are retaining our definition of what we understand as \(I_1\) and \(I_2\). In this case, because of the minus sign the bracketed term in the mixed Poynting vector is totally imaginary, and so its real part is zero and

\[
I = I_1 - I_2 \quad \text{(13)}
\]

Thus there is no interference term, and, as is well known, the waves can be considered as completely independent.

Now let us introduce absorption into our supporting medium. This implies that both refractive index and characteristic admittance will be complex, and, in our sign convention, of the forms \((n - ik)\) and \(y = (n - ik)\).

There is little effect on the counter-propagating waves expressed in (10). Each field decays as \(\exp(-2\pi k z / \lambda)\) and so if \(I_1\) and \(I_2\) represent the irradiances at \(z = 0\), then, writing \(\alpha\), the absorption coefficient for \(4\pi k / \lambda\), the result becomes:

\[
I(z) = I_1 e^{-\alpha z} + I_2 e^{\alpha z} + 2\sqrt{I_1 e^{-\alpha z} I_2 e^{\alpha z}} \cos \phi
\]

which is unchanged from (10) except for the exponential decay.

A major change is found in the counter-propagating waves. The electric fields of the two waves are given by:

\[
\text{Wave 1: } E_1 = E_0 \exp\left[\left(\alpha z - \left(\alpha z\right)\right)\right] = E_0 \exp\left[\frac{\alpha z}{2}\right] \exp\left[\left(\alpha z - \alpha z\right)\right]
\]

\[
\text{Wave 2: } E_2 = E_0 \exp\left[\left(\alpha z + \left(\alpha z\right)\right)\right] = E_0 \exp\left[\frac{\alpha z}{2}\right] \exp\left[\left(\alpha z + \alpha z\right)\right]
\]

where \(\alpha\) is real. By expressing \(H\) immediately in terms of \(E\) we will avoid the complication of having to calculate the phase difference between them. Now

\[
I = I_1 e^{-\alpha z} - I_2 e^{\alpha z} + \frac{1}{2} \text{Re}\left[\left(\alpha z + \alpha z\right)\right]\]

\[
= I_1 e^{-\alpha z} - I_2 e^{\alpha z} + \frac{1}{2} \text{Re}\left[\left(\alpha z - \alpha z\right)\right]
\]

The third term, the mixed Poynting vector, is now

\[
\frac{1}{2} \alpha z \text{Re}\left[\left(\alpha z\right)\right] - \exp\left[\left(\alpha z\right)\right] - \exp\left[-\left(\alpha z\right)\right]
\]

\[
= \frac{1}{2} \alpha z \text{Re}\left[\left(\alpha z\right)\right] - \exp\left[\left(\alpha z\right)\right] - \exp\left[-\left(\alpha z\right)\right]
\]

Thus the mixed Poynting vector is essentially a general interference term, similar to that in (10) and (14). It demonstrates that counter-propagating waves can actually interfere provided there is absorption present and prevents us from separating the irradiances of the two waves as in a medium free from absorption. Curiously, unlike the interference term for parallel waves in an absorbing medium, its amplitude of fluctuation remains constant with \(z\).

In all of this it is important to note that the strange behavior is confined to the irradiances. There is no coupling between the amplitudes, neither electric nor magnetic. There will, of course, be standing waves, when the propagation is not completely parallel, but standing waves in themselves do not affect the flow of energy. It must also be emphasized that there is no ambiguity or mystery. The effect is well understood within our normal electromagnetic theory, although it was not always so.

The Reflectance and Transmittance Problem

Reflectance and transmittance quantify the splitting at a surface of the power density of an incident wave into a reflected and transmitted portion. For infinite plane waves they are defined as

\[
R = \frac{I_{\text{inc}}}{I_{\text{inc}}} \quad T = \frac{I_{\text{trans}}}{I_{\text{inc}}}
\]

where the suffix \(z\) indicates that the irradiances are the normal components and the suffixes \(R\) and \(T\) indicate the reflected, transmitted and incident components. Let us continue to limit ourselves to normal incidence but we will permit both the admittance of the incident medium, \(y_0\), and the surface admittance of the front surface of the system, \(Y\), to be complex. However this elaboration does not affect our amplitude coefficients. The important one from the present point of view is \(\rho\), the amplitude reflection coefficient that will usually be complex.

\[
\rho = \left|\rho\right| e^{\rho} = \frac{y_0 - Y}{y_0 + Y}
\]

where \(y_0\) in SI units is given by \((n - ik)\).

Now let us examine the fields at the surface. Let the surface be situated at \(z = 0\) so that it coincides with the \(x-y\) plane. Then the total tangential fields, \(E\) and \(H\), at the surface are given by:

\[
\begin{align*}
\mathbf{E} & = E_{0}^{\prime\prime} + E_{0}^{\prime} + \rho E_{0}^{\prime} \\
\mathbf{H} & = H_{0}^{\prime\prime} - H_{0}^{\prime} + y_{0} (E_{0}^{\prime\prime} - E_{0}^{\prime}) = \rho y_{0} (E_{0}^{\prime\prime} - E_{0}^{\prime})
\end{align*}
\]

Then the net flow into the surface is given by:

\[
\begin{align*}
\frac{1}{2} \text{Re}\left(\mathbf{E} \mathbf{H}^{\ast}\right) & = \frac{1}{2} \text{Re}\left(\left(E_{0}^{\prime\prime} + \rho E_{0}^{\prime}\right) \left(y_{0} E_{0}^{\prime\prime} - y_{0} \rho E_{0}^{\prime}\right)\right) \\
& = \rho y_{0} \left(1 - \rho y_{0}\right) I_{z} - \frac{k}{n} \sqrt{\rho y_{0}} I_{z} \sin \phi
\end{align*}
\]
The second term is the mixed Poynting vector at the interface and as the coordinate $z$ varies it will vary according to (17) as:

$$-2k \frac{\rho \rho^*}{n} I_2 \sin(2kz + \phi)$$

(22)

In the normal way, $k$ will be small compared with $n$ and so the mixed Poynting vector will be small. For the sake of rigor, however, we will try to avoid defining reflectance in an absorbing incident medium. We recall that the mixed Poynting vector is important only when we are dealing with the calculation of net irradiance. Inside an optical coating – and we return to this shortly – we calculate amplitudes, and any problem that could be caused by incorrect use of the vector is completely avoided. In the normal way, the vector intrudes when we wish to calculate the reflectance of the internal surfaces of an absorbing substrate for an incoherent multiple-beam calculation.

Our first observation is that with a small fluctuation in the position of the output surface, as in any substrate, there will be a tendency for the mixed Poynting vector to disappear in the manner of the interference term. There is an additional consideration when we take the absorption into account. Let us calculate the magnitude we might expect from an extinction coefficient in such a case. We can imagine that the substrate might be 1mm thick and that a sensible multiple beam calculation might require a second beam in the series after a double traversal of the substrate to have irradiance greater than 1% of that of the first. A pessimistic view of this double traversal of the substrate would give

$$\exp \left(-\frac{4\pi kd}{\lambda} \right) \geq 0.01$$

(23)

with $d$ at 2mm, and, say, at a wavelength of 1000nm. This would imply an extinction coefficient of, at most, 0.000183 corresponding to an absorption coefficient of 23.03cm$^{-1}$. We can assume that $n$ would not be less than unity and then the maximum possible magnitude of the mixed Poynting vector, from (22), would be 0.000366$I_2$. Now with an extinction coefficient of 0.000183 the irradiance in a travel of one wavelength would fall by a factor of $\exp(-4nk)$, that is 0.99770, a reduction in irradiance $I_2$ of 0.00230$I_2$, almost an order of magnitude greater than the greatest value of the mixed Poynting vector.

Thus, in those cases where a calculation of reflectance would be of significance, we should not worry about any intrusion of the mixed Poynting vector. There is no agreement on how we might define reflectance in an absorbing incident medium. We can either define it only in a medium free of absorption, when we will automatically set the extinction coefficient of the incident medium to zero, a common solution. Or we might use $\rho \rho^*$. The difference between these two cases is of the order of $k^2/n^2$. For the limiting extinction coefficient already calculated this would be 0.0000033% and the enormous magnitude of the corresponding absorption coefficient would, in any case, imply the impossibility of success in any reflectance measurement. The point that needs emphasis is that we never require the calculation of reflectance in a highly absorbing incident medium.

There are, however, some interesting questions, largely of a theoretical nature. We have situations in optical coating where an absorbing substrate is coated with an antireflection coating. For the side of incidence, where $\gamma_0$ is real, usually air, the requirement is clear. A dielectric antireflection coating should match the admittance of the absorbing material that is acting as incident medium by $y = (n - ik)$. Some mild rearrangement gives us

$$T = \frac{4\alpha (n^2 + k^2)}{n^2 (n + \alpha)^2 + (k - \beta)^2}$$

(25)

and this is maximized by $Y = (n - ik)^*$, the complex conjugate of the characteristic admittance [6]. It is easily demonstrated that a completely dielectric coating that matches $(n - ik)$ to $\gamma_0$ where $\gamma_0$ is real, will, if reversed, match $\gamma_0$ to $(n - ik)^*$. Thus the optimum antireflection coating on the rear surface of an absorbing slab of material will be identical to that on the front surface, provided both incident and emergent media are the same [6].

### Some Concluding Remarks

There have been many studies that treat the mixed Poynting vector, many reaching false conclusions. The first study that dealt completely correctly with the phenomenon appeared in 1950 [7]. In spite of this, there was continuing confusion and incorrect conclusions. The story, up to 1987, is briefly told by Fragstein [8] who was prompted to write a recent (at that time) report describing the mixed Poynting vector as a new effect. The difficulty that has seemed to many workers to be...
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insurmountable is that, in the presence of absorption, there is a lack of symmetry in the power flow when incident and emergent media are interchanged. This conflicts with what was thought to be the inviolate principle of reversibility. Reversibility, however, as we saw in an earlier article [9], requires that absorption be replaced by gain, a point that appeared completely to escape the workers. They tended to avoid the apparent violation of symmetry by somehow deciding either that the mixed Poynting vector did not represent a real power flow or that there was something deficient in Maxwell’s Equations.

Then there has been some misunderstanding that normal thin-film theory is somehow unable correctly to include the vector, leading to incorrect absorption results. Normal thin-film matrix theory is completely correct. Since it deals with the total tangential fields at the various interfaces of the system, the Poynting vector calculated from these fields gives the correct net power flows and gives correct assessment of any power that is lost. It includes all effects of the mixed Poynting vector completely automatically, even including the tricky inhomogeneous waves that occur at oblique incidence in absorbing media. The operation of the perfect absorbers, the designs of which were described in [9], depends entirely on the mixed Poynting vector.

References

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