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Contributed Original Article

Gain in Optical Coatings: Part 2
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Introduction
This is the second part of the gain in optical coatings article. In the first part [1] we were mainly concerned with semi-infinite media and the stability of solutions. Here we will be much more concerned with finite films. There is a lot of ground to cover in this second part and so we will assume knowledge of a number of results and techniques explained in detail in earlier articles in this series. Brief descriptions and appropriate references are included in the text.

We recall that we are using our normal optical thin-film conventions where optical constants (n-ik), both n and k being positive, indicate an absorbing material and (n+ik), one with gain. The normal incidence admittance locus of a thin film with gain is a spiral that opens out while in the right-hand part of the complex plane, but contracts as soon as the locus enters the left-hand side. Provided the locus is long enough, that is the layer thick enough, it eventually terminates at the point -(n+ik).

At oblique incidence the tilted admittances have to be used, but the behavior of the locus is similar except that, as for layers without gain, the locus direction is reversed for p-polarization beyond the critical angle.

A semi-infinite gain medium that is completely homogeneous affords no opportunity for the spiral to open and the corresponding locus is simply a point that sits on (n+ik) or the appropriate tilted admittance yielding a reflectance that is identical to that of a lossy medium with identical absolute magnitudes of n and k. However, any departure from complete perfect homogeneity and the spiral opens out to give an ultimate surface admittance of -(n+ik) and a reflectance that is the inverse of the first case. The first solution can be classified as unstable and the second stable.

We now deal with the situation where the film with gain is terminated with a discontinuity forcing a stable, unambiguous solution.

Reflectance and Transmittance
Reflectance and transmittance are calculated in the normal way, but because of the amplification, the usual rule that reflectance plus transmittance can not be greater than unity (or 100%) no longer applies. Also the electric field amplitude is no longer inversely proportional to the square root of the real part of surface admittance as it is for dielectric materials. However, the isoreflectance circles in the admittance diagram are unchanged and can also be added to the left-hand part of the diagram. There, the circles are nested about the point y0 and the reflectance increases with reducing radius until the reflectance at the point -y0 becomes infinite. Of course, in practice, gain limiting prevents the reflectance from rising to such levels but in this simple analysis we do not include gain limiting.

In the vicinity of -y0 the reflectance is exceptionally high and varies rapidly with the terminal admittance of the locus. Film thicknesses that have a locus terminating close to -y0 will therefore show very sharp and significant oscillations with wavelength, angle of incidence, or thickness. An example is shown in Figure 1 where, as with some of the other figures the gain has been set to a high level to simplify the result. Each peak represents another turn of the admittance spiral. The increments in thickness represent halfwaves. Transmittance, as always, is a little more difficult to visualize but, as with absorbing systems, we get some help from the potential absorptance of an increment, d, in physical thickness. Equation (15) in [2] is:

\[ A = \frac{4\pi nkd}{\lambda} \frac{1}{\text{Re}(Y_e)} \]  

where Ye is the surface admittance at the rear of the slice of thickness d. We can think of this, if positive, as the fractional decrease in flux towards the rear of the coating, or, if negative, the fractional increase. In our present case k is negative. Thus, as long as the locus is to the right of the imaginary axis, Re(Ye) will be positive, and the value of A negative. On the right-hand side of the imaginary axis, therefore, the net irradiance will rise towards the rear surface. In the left-hand side of the imaginary axis, Re(Ye) becomes negative making A positive, and now the net irradiance rises towards the front of the coating. The variations in Re(Ye) cause oscillations to appear in the outputs with a period roughly of a halfwave.

The film used in the calculations of Figure 1 and Figure 2 reaches the imaginary axis at a thickness of very roughly 2500nm. We can therefore expect the transmittance to exceed the reflectance until the thickness reaches, again very roughly, 5000nm. From there on the reflectance will tend to be larger. As the film becomes much thicker and the reflectance stabilizes, that fraction of the film associated with gain increasing towards the rear becomes smaller and smaller, and the transmittance therefore, falls.

Figure 1. Oscillations in normal incidence reflectance from a film of optical constants 1.00+0.02on glass in an incident medium of air as the thickness increases. The wavelength is 1000nm. The level is virtually the stable result when the physical thickness reaches 16000nm.

Figure 2. The transmittance of the film of Figure 1. Note the drop in transmittance as the layer thickness increases towards 16000nm.
Previous Work

Most of the earlier workers appeared to prefer modifications of multiple-beam solutions. Some struggled with what happens at the critical angle and introduced, using various arguments, largely unconvincing, a discontinuity in the optical constants that presented the stable solution above the critical angle, happily corresponding to what is found in practice. Part one of this account discussed that situation and how it essentially accidentally gives a valid result. The most useful account from these earlier studies is that of Callary and Carniglia [3]. Callary and Carniglia also used a multiple beam approach. However they began with a quite finite layer thickness and then, to treat gain in a semi-infinite emergent medium, extended the thickness of their film to infinity. The discontinuity at the termination of this film then led immediately to the stable solutions both above and below critical.

We can compare their results just below critical, their Figure 5, with calculations based on the usual matrix method. This is shown in Figure 3. Their values of optical constants were chosen to mimic some experimental results of Lebedev, Volkov and Kogan [4].

Amplifier

Figure 4. The electric field through the amplifying layer. Since the matching at the rear surface inhibits the opening of the spiral locus, the field shows no signs of any standing wave and, therefore, of any counterpropagating wave. Air incident medium, film 10000nm of $(1.45+i0.01)$ perfectly matched at rear surface (at right). The front surface has no coating and so the incident medium (extreme left) does exhibit a small standing wave. A suitable conventional antireflection coating would remove this.

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A layer with gain, like an optical fiber, can act as an amplifier. However, for this application the matching of the rear surface has to be arranged to avoid spurious oscillations. The antireflection of the surface of an absorbing material has already been studied [5] using maximum throughput as the optimum criterion. The transmittance of the rear surface was shown to be given by:

\[
T = \frac{4y_0y_\ell \text{Re}(Y)}{\text{Re}(y_\ell)[(y_\ell + Y)(y_\ell + Y)^*]} \quad (2)
\]

where \(y_\ell\) is the absorbing incident medium admittance and \(Y\) is the admittance presented by the structure at the rear surface. The transmittance is then maximized by a value of \(Y\) that is the complex conjugate of the incident admittance \(y_\ell\). If the direction of the light is reversed then this coating will be an exact antireflection coating for the absorbing material. Such a definition, although yielding maximum throughput, is not useful for an amplifier because it would cause the admittance locus to open out, implying a standing wave and, therefore, a counter-propagating wave. To prevent an opening spiral, the exit admittance for the gain medium must equal the characteristic admittance of the gain medium, not its complex conjugate. Note that the matching applies to the given value of gain only. A gain change will disturb the matching, and the spiral will begin to open.

Metal Loci

Metal loci have already been described [6, 7]. A perfect metal, that is one with zero \(n\) and finite \(k\), has circular loci, centered on the real axis and cutting it in two points with \(-k^2\) as the product of their admittances. All the circles intersect at the points \(k\) and \(-k\) on the imaginary axis. Normally only that part of each circle that is in the right-hand part of the complex plane is permitted. All those parts of the circles are described clockwise. When the refractive index \(n\) is small, but not zero, then it is as if the system of circles were rotated so that the points \(ik\) and \(-ik\) become \(-(n - ik)\) and \((n - ik)\) respectively. The loci depart very slightly from perfect circles but as long as \(n\) is small the departure from perfect circles is miniscule. All this has already been explained.

What is now new is the presence of gain in the system that opens up the left-hand side of the complex plane. Now the full circles are permitted but those parts of the circles to the left of the line joining the points \(-(n - ik)\) and \((n - ik)\) are described counterclockwise rather than clockwise. This is illustrated in Figure 5. At oblique incidence, the points \(-(n - ik)\) and \((n - ik)\) move to the appropriate tilted admittances for the metal.

Surface Plasmons

A metal-dielectric boundary can, under certain conditions, support a propagating electromagnetic disturbance that is pinned to the surface and decays exponentially into both materials. The phenomenon, known as a surface plasmon, has been discussed recently in the Bulletin [8, 9]. There is considerable current interest in surface plasmons, but in many applications their inherent loss is a serious limitation and it has been proposed, and demonstrated, that the addition of gain to the dielectric medium next to the metal can compensate the loss [10-12]. The geometry used to demonstrate the effect is the Kretschmann coupling geometry shown in Figure 6. In the normal way, the dielectric material bounding the outer surface of the metal is infinite in extent and lower in index than the material of the prism so that a critical angle exists. The sharp plasmon resonance, where the incident light actually couples into the plasmon, then occurs beyond the critical angle.

We take as an example, Figure 6, a silver film of thickness 54.43nm on a BK 7 glass prism. The wavelength is 632.8 nm and the optical constants at that wavelength are Glass \((n = 1.5151)\) and silver \((n - ik = 0.0666 - 4i0.0452)\). The final semi-infinite medium is the air and since that is beyond the critical angle its tilted admittance is for \(p\)-polarization positive imaginary. Then the silver locus starts on the imaginary axis and loops round towards the real axis. At the correct angle of incidence the metal layer will intersect the real axis at the point corresponding to the incident medium admittance. Provided the thickness is such that the locus terminates there, the \(p\)-reflectance will then be zero. Because the parameters vary rapidly with angle of incidence, the reflectance shows a very narrow dip as a function of angle, the surface plasmon resonance. This is illustrated in Figure 7 and Figure 8. This was also described earlier in [6].
Figure 7. The $p$-admittance of the air emergent medium is purely imaginary beyond the critical angle and falls down the positive limb of the imaginary axis with increasing angle. As it passes close to the admittance of the metal, the metal locus, if of the appropriate length, then passes through the admittance of the glass incident medium, as shown. This leads to the usual plasmon resonance, shown in Figure 8.

Now, if we introduce a miniscule gain into the dielectric emergent medium and there are no perceptible fluctuations in it so that the admittance presented to the metal outer surface is the tilted $p$-admittance of the gain material, it will be in the fourth quadrant of the complex plane (Figure 1 of [1]). There will then be no surface plasmon resonance dip visible in the reflected beam in the prism [13] because there is no way that the subsequent metal locus can connect with the incident medium point on the real axis. This, however, is the unstable solution. Any fluctuation of any kind sufficiently deep in the semi-infinite medium and suddenly the stable solution will appear in the second quadrant of the complex plane (Figure 10 of [1]). When the gain is very small, this will...
normally be very close to the imaginary axis and then the resonance will be largely restored although the dip may not be quite as deep because the starting point of the locus has been pulled off the imaginary axis just into the second quadrant.

![Figure 9. The Kretschmann arrangement for the measurement of the gain-assisted surface plasmon (After [11]).](image)

Now let us remove the gain from the semi-infinite emergent medium and introduce a finite layer between it and the metal, and arrange for this new layer to exhibit gain, Figure 9. We also make sure that this layer will be operating beyond critical by assigning it the optical constants \((n - ik = 1.00 + i0.0012)\) so that its tilted admittance will be quite similar in magnitude to that of the emergent air medium except that it will have a very small positive real part and its imaginary part will be negative. The air fixes the starting point for the locus on the positive limb of the imaginary axis, just as in Figure 7. The gain layer will start at that point and will enter the left-hand side of the complex plane and move towards the terminal point that is given by \(-\eta_p\), where \(\eta_p\) is its tilted admittance. What happens to the remainder of the locus, depends on the exact value of \(\eta_p\) and on the phase thickness of the layer. For vanishingly small thickness and/or small gain, the terminal point of the locus will be just to the left of the imaginary axis and well to the right of the line joining the positive and negative metal admittances illustrated in Figure 5. Then the subsequent locus and the resulting plasmon resonance will be hardly affected. As the thickness of the layer is increased, however, the terminal point of the gain layer will move away from the imaginary axis towards the line marking the metal admittances, we can call this the metal line, and since this will pull the starting point of the metal layer with it, the metal termination will move away from the incident medium admittance and the minimum reflectance will rise. As the gain layer termination approaches the metal line, the metal layer will also curve downwards more towards the imaginary axis, further increasing the minimum reflectance. Provided the gain is large enough, eventually the termination of the gain layer locus will reach the metal line and pass through it. When that happens, the metal locus will suddenly flip to the other side of the metal line, rotating counterclockwise, and the reflectance will exceed 100%. Since the phase thickness of the gain layer will decrease with increasing angle of incidence, the higher reflectance will tend, at least at the start, to be displaced slightly to smaller angles compared with the original resonance because increasing angle will tend to pull the gain layer termination back across the metal line, flipping the metal locus back to clockwise rotation. This is illustrated in Figure 10 and Figure 11.

![Figure 10. The resonance calculated for three different thicknesses of the gain layer with optical constants \((1.00 + i0.0012)\).](image)

The situation becomes a little more complicated if the index of the gain layer is higher than that of the emergent medium. Then an increase in thickness of the gain layer will tend to move the end point up the imaginary axis as well as slightly into the left-hand part of the complex plane. This will increase the resonance angle, the increase being more rapid, the higher the index. The effect is illustrated in Figure 12. Many of the published results avoid this displacement by varying the gain rather than the layer thickness.

![Figure 11. The p-reflectance as a function of incident angle and of thickness of the gain layer with optical constants \((1.00 + i0.0012)\).](image)

![Figure 12. A change in the optical constants of the gain layer to \((1.45 + i0.01)\) causes the resonance to shift considerably in angle with changing thickness.](image)
Conclusion

Much of the behavior described has been confirmed by experiment. Although they may appear obscure and involved, the effects can be explained and calculated by our normal thin-film theory. Optical coatings is far from a static subject.

References