**Gain in Optical Coatings: Part 1**

Angus Macleod  
Thin Film Center, Inc., Tucson, AZ

**Introduction**

Gain in optical coatings and surrounding media is an interesting topic that appears from time to time in the literature. So far there are few definitive experiments and so it is largely a theoretical topic in which opinions sometimes differ. One of the aspects that exercise contributors is the existence, or not, of what is usually termed evanescent gain. Does a semi-infinite gain medium, illuminated beyond the critical angle, yield a reflectance greater than 100%? Whatever the opinions and the theoretical arguments, optical fibers with gain in their cladding do exhibit signal amplification. But can we really treat their cladding as an example of a semi-infinite medium?

This article is written largely because, at least to the author, it is a fascinating subject, but also included is the perhaps faint hope that understanding of the effects in thin films might contribute positively to the subject. It echoes and extends some material already published [1-3].

We will use our normal optical thin-film conventions in what follows.

**A Simple Gain Medium**

Optical gain can be simulated by reversing the sign of $k$ so that the optical constants become $(n + ik)$. In any real case $k$ will be quite small. However, so that we can more clearly see what is happening, in many of the examples we will adopt a quite large value of $k$.

Our normal thin-film theory is completely valid for $(n + ik)$. In the first instance we will assume completely homogeneous material with no fluctuations in the values of either $n$ or $k$. This will assist in keeping the theory straightforward. But we do note that in any practical case there will be variations, especially in $k$. We will see shortly that these can exert an important influence on the results.

First of all, the reflectance of a simple surface between a semi-infinite incident medium (lossless) and a semi-infinite gain medium is given by the usual expression:

$$R = \left[\frac{(n_0 - n)^2 + k^2}{(n_0 + n)^2 + k^2}\right]$$  \hspace{1cm} (1)

and, clearly, the sign of $k$ exerts no influence on reflectance, which is less than 100%.

Oblique incidence is a little more complicated. We follow the derivation for metals in an earlier article [4]. Again, as in that article, so that the isoreflectance circles remain in their normal incidence position, we normalize the tilted admittances by multiplying the $p$ and dividing the $s$-values by $\cos \delta_0$, where $\delta_0$ is the angle of incidence in the incident medium. Thus:

$$\eta_p = \frac{y \cos \delta_0}{\cos \vartheta}$$  \hspace{1cm} and  \hspace{1cm} $$\eta_s = \frac{y \cos \vartheta}{\cos \delta_0}$$  \hspace{1cm} (2)

where $\eta$ is the tilted admittance, $\vartheta$ the angle in the appropriate material and $y = (n + ik)$ free space units, the material characteristic admittance that may represent absorption or gain. These expressions apply without difficulty to dielectric materials, but a finite value of $k$, either positive or negative, presents complications in the evaluation of $\vartheta$ because Snell’s Law becomes:

$$n_0 \sin \vartheta_0 = (n + ik) \sin \vartheta = \ldots$$  \hspace{1cm} (3)

and $\vartheta$ is now complex. Fortunately, as for the metals, we do not actually need $\vartheta$. The required expression is $(n + ik) \cos \vartheta$ and it is possible to derive this while avoiding any complex angle problem. From (2),

$$\eta_t \cos \delta_0 = \eta_s \cos \delta = (n + ik) \left[ \pm \left(1 - \sin^2 \vartheta \right)^{1/2} \right]$$  \hspace{1cm} (4)

$= \pm \left(n^2 - k^2 - n_0^2 \sin^2 \delta_0 + i2nk\right)^{1/2}$

For the physical argument, it is convenient to think of $k$ as vanishingly small although always finite. There are two roots, one with positive and one with negative real part. In both cases we can see that the positive real part must represent the correct solution, otherwise the light will be travelling in the inverse direction. With $(n + ik)$ the two roots are in the second and fourth quadrants and so the fourth is correct. With $(n + ik)$ the solutions are in the first and third quadrants with the first quadrant, therefore, as the physically correct one. As $k$ tends to zero both solutions tend to the correct completely dielectric solution. Siegman [5] terms the positive root, that is the physical solution, Fresnel, and the negative, Lensenf. We shall see shortly the significance of the latter solution.

For our gain medium, then,

$$\eta_s = \frac{n^2 - k^2 - n_0^2 \sin^2 \delta_0 + i2nk}{\cos \delta_0}$$  \hspace{1cm} (5)

in the first quadrant, and

$$\eta_p = \frac{\eta_s}{\eta_t} = \frac{(n + ik)^2}{\eta_t}$$  \hspace{1cm} (6)

the quadrant of $\eta_p$ being unambiguously assigned by the expression (6). Then the expression we will shortly need for $\delta$, the phase thickness of the layer is derived from (4) as:

$$\delta = \frac{2\pi}{\lambda} \left(n^2 - k^2 - n_0^2 \sin^2 \delta_0 + i2nk\right)^{1/2}$$  \hspace{1cm} (7)

also in the first quadrant. The reflectance is given by the usual expression:

$$R = \left[\frac{\eta_p - \eta}{\eta_t + \eta}\right] \left[\frac{\eta_s - \eta}{\eta_t + \eta}\right]$$  \hspace{1cm} (8)

**Internal Reflection**

Internal reflection implies an incident medium with refractive index greater than that of the emergent medium. If the two materials are completely dielectric, that is without loss, then there is a critical angle of incidence, given by $\arcsin(n/n_0)$, $n$ being the emergent index, above which the reflectance becomes 100%. The phenomenon is known as total internal reflection. Although, strictly, there is no critical angle as soon as the emergent medium index becomes complex, it is useful to continue to refer to a critical angle, calculated by setting the imaginary part to zero. The expressions and rules from the previous section apply equally well to incidence beyond or below critical.
Figure 1. Variation with angle of incidence of the tilted admittances of the gain medium $(1.0 + i0.01)$ with incident medium $n_0 = 1.52$. Note that these are the normalized admittances, and that retain constant the admittance of the incident medium.

We can plot the variation of the tilted admittances with angle in the complex plane, Figure 1. Note that nothing strange happens to the admittances at the critical angle. These curves lead straight to the reflectances plotted in Figure 2. Also shown in Figure 2 is the internal reflectance assuming air as the emergent medium with total internal reflectance existing beyond the critical angle, marked by the rapid almost vertical rise of the reflectance. Adding gain to the outer medium actually reduces the reflectance beyond the critical angle. Even with gain the reflectance is nowhere greater than 100%. In fact the reflectance of the surface with a gain of 0.01 is indistinguishable from that of a surface with an extinction coefficient of 0.01.

Figure 2. The zero gain curves show the usual plot of internal reflectance versus incidence. The broken lines show the corresponding reflectance when gain $(k = 0.01)$ is added to the outer medium. $n_0 = 1.52$, $n = 1.00$. continued on page 24
Gain in Optical Coatings: Part 1
continued from page 23

Figure 3. The phase shift on reflection (normal thin-film sign convention) is flipped by reflection in the horizontal axis whenever gain is involved.

There is, however, a curious phase behavior in reflection. The amplitude reflection coefficient is given by

$$\rho = \left| \rho \right| \exp (i \varphi) = \frac{\eta_0 - \eta}{\eta_0 + \eta}$$  \hspace{1cm} (9)$$

Since the sign of the imaginary part in this expression is inverted in the presence of gain there is a change in the quadrant of the phase shift on reflection, $\varphi$. This is illustrated in Figure 3. An earlier article explains how to assess the variation directly from Figure 1 [6].

All this is quite straightforward and shows the complete absence of gain in reflection either above or below the critical angle. Yet Table 1 suggests otherwise. Why is this and why the different findings? To answer this question we need to look closely at gain and its effect on the stability of the solutions.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Amplification below critical</th>
<th>Amplification above critical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Romanov and Shakhidzhanov [7]</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Callary and Carniglia [8]</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Willis, Schneider and Hagness [9]</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Siegman [5] (Fresnel)</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Siegman [5] (Lenserf)</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Ignatovich and Ignatovich [10]</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Gain and Stability

Let us start by reminding ourselves of the effects of absorption in a thin film.

As always, the linear nature of the interactions between electromagnetic waves and materials permits us to represent any arbitrary light input wave as a set, or spectrum, of plane, linearly polarized, harmonic components. The complex form of this simple component allows us to write the optical constants of an isotropic material as $(n - ik)$, where $n$ is the refractive index and $k$ the extinction coefficient. The extinction coefficient is a measure of the exponential decay of the wave in the presence of absorption in the material. We recall that, $(n - ik)$ with no units is the complex refractive index, but in the optical region and with units of the admittance of free space, or free space units, it is also the characteristic admittance, $y$, of the material.

We further recall that the admittance locus for a dielectric layer with zero extinction coefficient is a circle centered on the real axis and described clockwise, each complete rotation representing a half-wave. When a finite extinction coefficient is introduced, the locus becomes a reducing spiral that eventually ends on the point $(n - ik)$, the rate of convergence being greater the larger the value of $k$. Figure 4 shows a typical locus of an absorbing material. The extinction coefficient has deliberately been chosen rather large to make the convergence rapid. Clearly this is a completely stable unambiguous solution.

Now let us carry out a similar calculation but, this time, with gain rather than absorption in the coating material. Again so that we can readily see what is happening we use quite large gain represented by a $k$-value of 0.2. The locus is shown in Figure 5. Now the spiral, instead of shrinking, expands clockwise, eventually reaching the imaginary axis and passing through it. Although the locus may look in the figure as though it is passing through the origin, the point of intersection can be anywhere depending on the precise values involved. Once through the imaginary axis, the locus reverses so that it turns counterclockwise and shrinks. Provided the layer is sufficiently thick, it then converges and terminates on the point $-(n + ik)$.
as in the right-hand half of the complex plane. But the loci are now described counterclockwise. The isoreflectance circles are now nested about the point \(-y_0\), which indicates infinite reflectance rather than zero. Larger circles represent lower reflectance, with the ultimate circle being coincident with the imaginary axis and representing a reflectance of 100%, the minimum possible in the left-hand half of the complex plane.

Clearly a gain layer that is deposited over a substrate with exactly identical optical constants, \((n + ik)\), will have an admittance locus that is simply a point that coincides with the substrate admittance. This yields the reflectance in equation (1). However, should there be the slightest mismatch, the spiral will first open out, and then, if the layer is thick enough, converge on the point \((n + ik)\). Provided it has reached that point, the reflectance will be given by:

\[
R = \frac{(n_0 + n)^2 + k^2}{(n_0 - n)^2 + k^2}
\]

that is the inverse of the other value and clearly greater than 100%.

Thus there are two solutions. The first is the simple one, Figure 2, where the surface admittance of the semi-infinite gain medium is equal to its characteristic admittance. There must be no perturbations of any kind in the medium otherwise the spiral admittance locus will be triggered, and, if the path is long enough, will reach the reflectance given by equation (10). We therefore label the results of Figure 2 as unstable. The results of equation (10) will be reached only if the perturbation is great enough and the remaining film thick enough to reach the limit. We therefore label that solution the stable limit. Both solutions are shown in Figure 6.

The phase shift on reflection corresponding to the stable limit will now be flipped once again so that it now matches well the case of a dielectric or absorbing material. The curious phase jump of Figure 3 has now vanished, Figure 7.

In a sense, therefore, all the solutions of Table 1 are correct because they represent either the limiting stable or the unstable solutions. Also because the stable result for the phase shift on reflection is similar to the zero \(k\) case it is unlikely that the phase flip of Figure 3 could ever be measured. The phase flip depends on a homogeneous and semi-infinite gain medium, theoretically simple but practically virtually impossible. Nevertheless, it is of critical importance for calculation.

Figure 6. The two solutions for reflectance including both s- and p-polarization. The p-polarization peak corresponds to the Brewster angle, although because of the finite \(k\)-value it does not yield exactly zero reflectance. The results below 100% are unstable in the sense that any slight perturbation in the semi-infinite gain medium will cause the admittance spiral to begin to expand. If both the perturbation and the film thickness to the front surface are both great enough then the stable limit will be reached.

The phase shift on reflection corresponding to the stable limit will now be flipped once again so that it now matches well the case of a dielectric or absorbing material. The curious phase jump of Figure 3 has now vanished, Figure 7.

In a sense, therefore, all the solutions of Table 1 are correct because they represent either the limiting stable or the unstable solutions. Also because the stable result for the phase shift on reflection is similar to the zero \(k\) case it is unlikely that the phase flip of Figure 3 could ever be measured. The phase flip depends on a homogeneous and semi-infinite gain medium, theoretically simple but practically virtually impossible. Nevertheless, it is of critical importance for calculation.
In a number of the studies, the gain medium is not strictly semi-infinite but is terminated in some way that is intended to simulate an infinite extent. Willis, Schneider, and Hagness [9] terminated their gain medium with a perfectly matched layer designed to inhibit any reflection. In the Callary and Carniglia study, the calculations began with a quite thin layer (more about that in a subsequent article) and a large mismatch. Extending the thickness of the layer to infinity then gave, as would be expected, the stable limit.

Much of the remainder of the answer lies in Figure 1. Especially in the case of s-polarization, the values of the tilted admittance below critical are largely real with a quite small imaginary part. Beyond critical however, the imaginary part becomes large while it is the real part that is very small. As the real part of the admittance reduces while the imaginary part grows, the rate of expansion or contraction of the spiral increases. Eventually, when the real part is very small, the locus becomes virtually circular instead of a spiral (not unlike a metal [4]) and the convergence is then extremely fast. Oddly enough, the very high value of the imaginary part beyond critical implies much smaller amplification of reflectance than below critical.

**Why the Bias in Reported Results?**

Different theoretical approaches were used in the studies summarized in Table 1, and in other papers not cited. Many of the published studies treat s-polarization only. Amongst other considerations this happily avoids the rather large peak in amplification corresponding to the Brewster angle. [Rather like the critical angle, the Brewster condition strictly does not exist for an absorbing or gain medium, but we use the definition applying to purely dielectric media.] Although reflection with and without amplification is found both below and beyond the critical angle there is a definite tendency for more studies to show amplification above critical and none below. Typical of the results are those of Willis, Schneider and Hagness [9], illustrated in Figure 8 for their plane wave solution. What could be the reason for this?

We note that the two solutions for the surface admittance at the boundary with the incident medium have their origin in the positive and negative square roots of the expression in . Some of the studies use various arguments simply to pick one or the other of the two solutions. We have seen how both are valid, one representing an unstable and the other a stable solution. Siegman [5] actually considers both solutions and shows that the negative root (Siegman's Lensertf solution) implies amplification.

**Gain in Optical Coatings: Part 1**

continued from page 25

![Figure 7](image_url)  
*Figure 7. The phase shift on internal reflection when the admittance locus terminates at \((n + ik)\) (broken curves) is very similar to the phase shift for a semi-infinite material with optical constants \((n – ik)\) (full curves). Beyond the critical angle the curves with and without gain virtually coincide and there is no phase flip.*

![Figure 8](image_url)  
*Figure 8. Results for s-polarization from Willis, Schneider and Hagness [9] plotted against their plane wave solution. The Willis study modeled Gaussian pulses and as the beam waist increased, as would be expected, the solution tended to their plane wave result.*

![Figure 9](image_url)  
*Figure 9. The gain medium is 10 waves thick with optical constants \((1.00 + 0.01)\). This is terminated by a semi-infinite medium of constants \((1.00 + 0.00)\) to create a mismatch. Incident medium has \(n_0 = 1.52\) and incidence is 35°. The spirals have opened but are far from reaching the imaginary axis and a reflectance of 100%.*

![Figure 10](image_url)  
*Figure 10. Here, at 60°, the gain medium is 3 waves thick and the mismatch is between the optical constants \((1.00 + 0.01)\) and a semi-infinite medium of constants \((1.00 + 0.00999999)\) creating a much smaller mismatch than in Figure 9. The incident medium again has \(n_0 = 1.52\). The loci have reached the stable limit by 2.5 waves.*

We can illustrate this with a simple example. The system of Figure 9 has an incident angle of 35° in an in a medium of index 1.52. The emergent medium consists of a film of thickness ten waves and optical constants \((1.00 + 0.01)\), representing a very large gain so that, again, we may easily see what is happening. A mismatch is arranged at its...
outer surface with a semi-infinite dielectric medium of index 1.00 and zero gain and absorption, a fairly severe mismatch. Nevertheless the admittance loci of Figure 9 are far short of reaching the imaginary axis. To reach the stable limit a much greater thickness and, preferably, a much greater mismatch, is required, and, of course, with more realistic gain the spiral opening will be orders of magnitude slower. Figure 10, on the other hand shows the same system but at 60° incidence. The film in this case is now only three waves in thickness. The terminating mismatch, now with a semi-infinite medium of optical constants (1.00 + i0.00999999), is miniscule. Nevertheless, because of the dominating imaginary part already mentioned, the loci are almost circular and they have virtually reached their stable termination points at thicknesses of around 2.5 waves. Clearly, amplification is much more readily achieved beyond the critical angle.

The case of the fiber with gain in the cladding is clear. Reflection in the core is beyond critical. The cladding is finite with a considerable mismatch at its outer surface, and so the stable limit is the correct solution.

It is interesting to note that, as for lossless media, the p-polarized locus beyond the critical angle is reversed in direction. For the lossless materials, the p-polarized admittance loci beyond critical and in the right-hand half of the complex plane are described counterclockwise [11] all other loci being clockwise. Here in the left-hand half of the complex plane the directions are reversed, the p-polarized locus beyond critical being clockwise while all others are counterclockwise.

Conclusion

Simple optical coating theory comes to our aid when we are dealing with the problems of gain media and in particular what is termed evanescent gain. This article has concentrated on the properties of semi-infinite media. The next article will look more closely at finite thin films with gain.

References